QUANTUM LOGIC

and

THE SEMANTICS OF NATURAL LANGUAGES*

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* or "THE WARP HYPOTHESIS"

CORRIGENDA

Page 1: Next to last line should read; "On the other hand, while the second ....anomalies, classical logic clearly can not be......"

Last line, footnote deleted; I do not mean to imply that Godel's theorem is a mistake. Rather, while it may be a limitation we are forced to accept, inappropriate truth valuations are intolerable.

Page 2: Fifth line from the bottom, footnote deleted at "........ concepts of reality*"; i.e. conceptualization schemas.

Page 23: Line five; quotes should follow "that" and not precede it.

Page 25; Second line from the bottom, footnote deleted at "...two different levels*."; Bill Miller of Diablo Valley College suggested the following example -

"He walked over to the billiard table and swallowed the cue-ball."

- in which the meaning (overt) is changed by the addition of a single word, thus -

"He walked over to the billiard table and swallowed the cue-ball <u>again</u>."

Page 26: Last line; "...at different <u>levels can occur</u>."

Footnote 19 deleted: see Finkelstein, STC II.

NOTE: Throughout this paper examples appear which were used in the discussions found in <u>Everything That Linguists Have Always Wanted To Know About Logic</u>* by James McCawley, University of Chicago. It should therefore be awknowledged that, without the opportunity to read the book in its prepublication form, the arguments presented here would have been "up a theoretical creek without an example."

I. Introduction

Logic is more than the determination of logical truths; it must also be concerned with the semantic analysis of propositions if it is to be an applied science. It is generally assumed that natural language is capable of multiple expressions of equivalent semantic content. For both linguists and logicians, the structural relationships between semantically equivalent expressions is of fundamental importance and in fact the structural relationships between all meaningful linguistic utterances is of prime import. Until we find a satisfactory model of such structures, our understanding of both logic and linguistics will be sorely limited.

Finding linguistic universals and cataloging them under some ordering system such as generative grammars is one method of getting at the semantic substructure of propositions in linguistic utterances. Another method might be the application of classical logic to such propositions in the hope that some recurring and purely logical framework might make itself apparent. The trouble with the first of these possibilities is that it forces the data into a mold after first having filtered out anything that would not readily fit. That is, such a methodology will only result in a listing of those aspects of natural languages which happen to fit the ordering system. The items of real interest - the anomalies - will be passed over and hence can not be used to evaluate the efficiencey of the model. On the other hand, the second possibility has clearly been fruitful in pointing out anomalies. Classical logic clearly can not be construed as the logical substructure of linguistic utterances. Not only does classical logic remain incapable of providing a deductively rigorous framework for many statements (as Gödel[1] has shown), but it commits the truly undesirable and unnecessary error of forcing inappropriate truth valuations.[2] It is therefore safe to conclude that the model is inappropriate and that either the modification of the model or the use of a radically different model is needed in order to incorporate the observed anomalies (especially those relating to truth valuations) within a single framework.

Any new model which one might wish to propose must certainly retain the useful features of classical logic.[3] There is much which classical logic is able to predict correctly concerning the semantic (meaningful) elements of natural linguistic discourse and to completely disregard this fact would be foolish. However, the solution can not be so simple as to be a modification of the rules of inference of classical logic.[4] There is an alternative to the problem. We shall demand that under some given set of conditions, our new model or logical structure must reduce to that of classical logic and, specifically, that it shall do so precisely under those conditions in which classical logic as applied to linguistic utterances remains valid.

While such a task as the one described might well be an impossible one, we are not without clues as to the starting point in our search for a more inclusive and descriptive logic. The connection between language and concepts of reality is, admittedly, a complex and debatable issue.[5] Nevertheless, we can be quite certain that language is of an empirical nature;[6] that is, that language is not constant but changes according to the empirical reality of its users. In much the same way, logic itself may be of an empirical nature. Putnam (1969) has argued that this is precisely the case with the overthrow of classical physics and the resulting philosophical issues that have arisen with regard to quantum mechanics and general relativity. For example, consider the geometry of general relativity as compared to that of Euclid. In an Einsteinian geometry, two lines can be **parallel** and still converge. And yet classical logic would never admit to the validity of such a proposition.[7] Still, the Einsteinian geometry is fundamental to the notions of general relativity and general relativity itself is an empirical fact! As a second example, consider the problem of complementarity a posed by quantum mechanics. Classical logic predicts no problem in asserting that a particle has a position so-and-so and a momentum such-and-such. In fact, determinism demands infinite precision from both variables. Quantum mechanics empirically refutes that claim as does the geometry upon which it is theoretically founded.

Finkelstein (1969) has carried the analysis even further and argues that logic is explicitly dependent upon the geometry from which notions of reality are derived. In a later series of papers, he carries out the process of deriving the Space-Time Code; that is, the logic which underlies our knowledge of the world especially in as much as it is relevant to physics. The question that now arises is of critical importance. If natural language is subject to empiricism and if logic itself is subject to empiricism, then unless the constructs of reality underlying physics andthe use of natural language are quite different, is there any reason to assume that the logical substructure[6] of one should be different from that of the other?

Assuming that the logic of the empirical sciences is equivalent to that which underlies natural language and therefore meaningful linguistic utterances, it should be possible to find logical parallels between the two structures. The fact that anomalies result in both under a classical logic analysis does not lend support to the hypothesis under consideration. If, on the other hand, a single logical model were found which predicted the anomalies and therefore reduced them to standard propositions within the proposed framework, there would be sufficient reason to consider the hypothesis as a working theory and to investigate its implications further. Again there is a difficulty of primary importance. The fields of linguistics and the natural sciences use quite different meta-languages. When one speaks of a space-time point in physics for instance, to what does that object (logical object in fact) correspond in linguistics? What is the corresponding and parallel concept for a "geometry" with its componential concepts "topology" and "metric"?

The first task then in deriving a linguistic logic along the ideas outlined above will be to establish a meta-language through which consistent correlations between two areas of knowledge may be drawn. This task will be mediated to some extent by the second task - the choice of a hypothetical model of the logical substructure. We need not concern ourselves with the development of unneeded and superfluous vocabulary for the meta-language, but must establish a vocabulary sufficiently rich to insure an adequate treatment of the problems at hand. Finally, we shall be interested in noting specific examples of the explanatory power of the model. At the same time it might also be hoped that improvements in the model might be suggested by the investigation if the methodology is sound.

II. A First Model: Causal Quantum Logic

As a first approximation to the logical substructure, we seek a model which serves to describe as much of the empirical data as is currently possible. Specificly, we want to make use of that logic which serves to provide the foundation for much of physics in the hope that such a logic will be representative of the logic underlying the other natural sciences and, at the same time, will be exemplary of the more rigorous and (most important) more detailed of logical models based on empirical data. This logic is commonly known as <u>quantum logic</u> and was first explored in depth by von Neumann and Birkhoff (1936) in an attempt to provide both alternatives to the model for the algebra of attributes proposed by Boole[9] and a coherent logical framework for the seemingly <u>ad hoc</u> mathematics of quantum mechanics.

In recent years, Finkelstein (1969,1972) has refined quantum logic in an attempt to define the presumed geometry of space-time in such a way as to eliminate the contradictions between general relativity and quantum mechanics, satisfy the dilemma posed by Riemann in the 1800's,[10] and thereby unify physics. Although a later paper by Finkelstein (1974) posits a <u>relativistic quantum logic</u>, it will not be necessary to invoke this model for the purposes of the current paper.[11] Nor will we consider quantum logics generally, but rather we will restrict the model of interest to that which presupposes a causal ordering relation. The topology of a space must be based on ordered intervals with some ordering relation. Since we can arbitrarily restrict the discussion to those linguistic utterances which admit of a causal logic (not necessarily a classical logic).[12] we shall be interested in the topology of a space defined under a causal ordering relation - in fact, a causal linear ordering relation. It is the topology of this space - a geometry defined by a causal quantum logic - which we will use as the basis for the required meta-language.

To simplify matters greatly, quantum logic may be thought of as being a generalization of set theory as applied to logic. Quantum logic is a calculus of classes defined over a system whose abstract algebra is the quantities of that system.[13] For a quantum logic as defined by Finkelstein (1972), in general the set in a classical logic becomes an underlying linear Hilbert space, and a subset becomes a subspace. For example, the classical interpretation of inclusion - P is included in Q - is the subset inclusion while the corresponding quantum logic interpretation is the subspace inclusion. The formal definitions of the operations of quantum logic are tabulated in Table I.[14]

In order to derive the proper analogies in defining a meta-language, we still need to take a look at the restrictions which Finkelstein places upon such concepts as a space-time point, Hilbert space, causal ordering relation, etc. First of all, quantum physics is concerned with quantum objects. Secondly, we have had to recognize the fact that the answer to both of the questions posed by Riemann is an unqualified "No": matter is neither continuous nor is it discontinuous but composed of quanta. Following Finkelstein, we define a quantum object to be one whose class calculus is non-distributive and is the lattice of subspaces of a separable Hilbert space which defines an algebra of operators (Hermitian) on that Hilbert space. Finkelstein suggests that the necessary conditions on the structure of space-time are also satisfied by a quantum manifold. Such a geometry generates the concept of a point in space-time as an assembly with an intricate internal structure. Each point must have a precision memory unit, a logic unit, multiple-access input/output units, and in general might be characterized as a sort of computer.[15] It is important to remember that these units of the assembly are of an intrinsicly abstract nature. Points can have no mechanical properties attributed to them. In order to prevent confusion with other concepts of the space-time point, Finkelstein refers to this concept as a "digit" and to quantum sets of quantum digits as "words".[16] Hence the general task arises of "breaking the space-time code" and such is the theme of Finkelstein's work: discovering the rules for the generation of words which in the classical limit will give the causally structured geometry of space-time.

We take the logic which operates on such quantum digits to be isomorphic with the geometry of the space they define. In the case under investigation, we take the logic to be the algebra of operators defined by a class calculus of attributes which is a complete, orthocomplemented, modular lattice. This definition will be found to serve us well in constructing the meta-language in the following section. However, for the sake of consistency, we must also be concerned with concepts such as the metric.[17] Fundamentally, a metric can be defined on a dimension function for lattices. We therefore include the restriction that our lattice admit a nontrivial, countably additive dimension function.[18] Finally, we say that a digital assembly will be regarded as "later" than some other assembly if and only if it is greater.[19]

Last but not least there is the useful terminology of "systems". We must define all operators over some system. A system is given by an algebra with adjoint operation and with complex numbers; i.e. the quantities of the system. Table I gives the operations of quantum logic for simple systems, compound systems (especially dealing with binary relations and the generation of compound systems from simple systems), and complex systems (the generation of complex systems from simple ones).

While there will be the need to introduce new terminology as the analysis of linguistic logic developes, the concepts defined here are those which are immediately necessary. As new terminology is introduced, we will also need to refer to the original context and theterminology already established will be sufficient for that purpose as well.

III. Establishing The Meta-Language

The particular branch of linguistics with which we are presently concerned is that of semantics. In fact, it might be said that a primary concern of this paper is to make a serious contribution to the systematic analysis of meaning with regard to the study of natural languages. We have chosen to attack the problem of developing such a systematic analysis of natural language meaning through the synthesis of a model whereby the logical substructure of natural languages might be more readily revealed.

Classical applications of logic have interpreted the proposition as a set and, in general, the meaning of a statement has been defined in terms of set theory. For example, consider the statement "That is a red flower." We might simplify this grammatical structure to a form more conducive to set theoretical interpretations by writing it as "Red, flower x" where the "x" represents a variable. Such a statement refers to that set which is the adjunction of the set of all red objects with the set of all objects having the properties of flowers. We now seek to quantize this process of interpretation of meaning in the hope that it will allow us to more closely approximate the meaning of utterances.[20]

First, note that the term "flower" is ambiguous in the sense that given some set of defining properties of "flowers", we may not strictly determine how many (or which) of these properties may be excluded without the set being defined as something other than a flower. That is, we have no clear definition of the closure of the set. Second, we must also recognize that the set is ambiguous in the sense that the domain over which the set "flower" is to be used is undefined. For instance, if the statement under consideration were uttered by a judge at a floral show, one might well expect the domain to be different than that which would be intended by, say, a stuffy and severely academic lexicographer who was defining the term for the pilot of a UFO.[21]

While we may not be able to do away with ambiguities, the very least one can demand of a systematic analysis of natural languages (which just so happen to be of an ambiguous nature) is that the ambiguities be given some sort of formalism. Let us consider for a moment the first form of ambiguity demonstrated here. We made use of a very abstract term - that of "properties". We now define this term.

The term <u>property</u> will be used to refer to an abstract assembly of <u>quantum units of meaning</u>.

In turning to the second form of ambiguity, we note that the key term here is the concept of a domain. This concept presents a slightly more difficult problem in attempting to state a definition which is even temporarily adequate. Therefore we will attempt to present a definition from other, perhaps more familiar concepts.

Rescher (1971) has defined the process of temporal realization $R_t(A)$ to be that in which the proposition A is said to be <u>realized</u> at time t. A similar process of spatial realization may be written as $R_x(A)$, to be read as "A is realized at the position x." Generalizing this notion to a spacetime set or <u>subspace</u> as in the quantum logic of Finkelstein, we may write $R_{\underline{a}}(A)$ to be read as "A is realized in the subspace <u>a</u>". Rescher also demonstrates that the process of realization (let us say the "realization operator") along with the ordering operator U suffice to form a weakly complete axiomatic basis. We now define the elements of the subspace a to be ordered according to some ordering relation U. Finally, let us generalize the so-called realization operator. We say that A is realized in the domain[22] a with ordering relation U if, for some subspace of constraints A' (for instance, a set of percepts or a reality construct), A' is minimal with respect to A, A is included in the subspace A', and A is defined only over the ordered subspace a. At this point, observe that the process of realization as used in temporal or tense logic by Rescher is taken here as presupposing a sort of refernce subspace (set) or world-view. We can not take this concept as being general since this subspace is presumed to be of a universal nature. We reject such a notion categorically. Different speakers of the same language have different world-views or reality constructs as do those with different languages. Indeed, there is no reason to assume that the same speaker of a given language at two different moments of time can be taken as having the same reality construct. A similar view is echoed by Birkhoff (1940) in a general criticism of the notion of a "universe" in the sense of Boole.

In the preceding paragraph, continual reference has been made to the concept of a subspace with "elements" formed according to some ordering relation U. We have been avoiding a formal definition of these terms in our meta-language until now in order to maintain clarity while other, more tedious concepts were being developed. The concept of a subspace necessarily implies that of a space. We define the relation of a logical substructure of natural languages to its isomorphic geometry as satisfying the following conditions: first, a semantic space which is a complete, orthocomplemented, modular, non-distributive lattice; second, borrowing a term in part from Finkelstein, a semantic digit which is a quantum unit of meaning having an intrinsically abstract and complex internal structure; and third, an assembly which is any quantum set of quantum digits such as a word (as in digit), property, attribute, or tagmeme.

Having defined these basic elements of the meta-language, we are now equipped with sufficient terminology to build more complex concepts which will be necessary to the macroscopic application of a quantum logic to meaningful linguistic utterances. Note that previously it was stated that all operators are defined over a system with an algebra of classes. These operators are said to operate on propositions. Before such a concept can be meaningful, it must be more explicitly defined. We therefore define the propositional variables (for example P, Q, etc.) to refer to an assembly which defines a subspace. Under these conditions, we take the orthocomplement, conjunction, and adjunction to be subspaces and not, as classicly assumed, propositions. However, we take the inclusion relation to define a proposition as a necessary but not a sufficient condition. We complete the necessary conditions that an explicit inclusion relation be a propositon by demanding the associated eigenvalue equation be non-degenerate.[23] That is, there must be a unique meaning associated with the propositional variables and with the binary inclusion relation associated with any two of them.

While a statement or linguistic utterance may well be grammatically acceptable and logically complete, it still need not necessarily qualify as a proposition. In order to distinguish the two,[24] we postulate the notion of propositional completeness. Given a subspace of semantic digits, which under some ordering relation combine to form an abstract assembly recognizable at the linguistic level, this higher ordered assembly or complex need not have an identifiable meaning (macroscopicly). On the other hand, the addition (or perhaps even the subtraction) of a single quantum element of meaning to (from) the subspace may suffice to bring about such a meaning. An utterance which has the property of being without identifiable meaning or which is capable of multiple interpretations is thus said to be propositionally incomplete or, on the macroscopic level, is said to be ambiguous. For a given domain and ordering relation, we claim that the closure of the subspace is unique. That is, the subspace of digits which define the closure for a given meaning is unique. We now explicitly point out that the semantic digits need not be of a purely linguistic nature. For example, the possible closures for the statement "The flower is red" that were given above are not necessarily of a linguistic nature. The information referred to could very well have been discovered in the form of visual inspection by a third party observer. In general then, we point out that a semantic digit may be perceived through any mode of perception or memory. A statement which is propositionally incomplete is therefore generally said to be context sensitive.[25]

In order to distinguish between statements which are incomplete in the sense of having no macroscopicly assignable meaning versus those which have many (which is sometimes the intention of the speaker as with puns, poetry, etc.), we shall say that a statement which admits to more than one meaningful interpretation is a superproposition. A superproposition may be decomposed into a set of linearly superimposed propositions. A superproposition is not subject to logical analysis as the domain of the statement is, shall we say, "subject to change without notice". Therefore, the proper procedure is the careful and explicit stating of the domain, thus insuring the consideration of one proposition at a time. Generally, one must place the proposition in the format defined earlier, thereby making explicit both the domain and the subspace of constraints (context). We shall refer to the multiple propositions which comprise a superproposition as its component propositions. At the level of articulation, a component proposition is identifiable as a syntagma.

Often a propositionally incomplete statement will occur on the macroscopic level which is not a superproposition, but which will admit to a meaning interpretation given some assumed subspace of constraints. Generally, one tends to make such assumptions in a way which minimizes the relative size of the subspace of constraints while at the same time generating a meaning assignment. Such a subspace of constraints is commonly known as an implicit premise, associated proposition, or presupposition.

We now turn to the final consideration in our meta-language, that of truth valuations. Truth valuations must be based on something other than arbitrary assignments of "true" and "false". We will take a truth valuation here as a subspace comparison. To say then that a proposition is "true" is to say that the semantic subspace which is isomorphic to that proposition is included in the subspace which is isomorphic to the reality construct which is defined over the same domain (i.e. the subspace of constraints). If a proposition is not "true", it is said to be "false". Note here that the reality construct is an instantaneous semantic (meaningful) object which is dependent upon: a) the belief system, b) the ordering relation (syntax?), and c) other constraining factors given both by concurrent and prior observation (percepts).

While the forgoing is the assignment of truth values and points out their empirical nature, there is another sense in which we are concerned with truth valuations. Essentially, we have to distinguish between that which is logically "true" and that which is "realized" in the sense of Rescher. For example, a false antecedent has nothing to say about the realization of the consequent nor of the implication itself. It merely states that the entire implication must be logically "true". Logical truths have absolutely nothing to do with either primary beliefs or observations once the logical substructure has been defined. They, instead, predict that the conclusion is realized in the given domain. That is, they are inferences based on the topology of a semantic space which is presumed to be locally homogeneous.

We say that a truth valuation of a proposition A is objective betweeen two or more individuals if they share a common subspace of constraints A' defined on a relevant domain a. A relevant domain is one which contains the domains of the proposition A and is minimal with respect to this property.

THEOREM: An objective truth valuation is unique.[27]

We call a subspace of constraints A' of a single individual a belief system. Then, a truth valuation with respect to a belief system is a belief valuation.

IV. Quantum Linguistic Logic: Examples

We have attempted in the previous section to provide a definition of the objects of logic - propositions - that will conform to the ready application of the rules of quantum logic and will, in fact, be consistent with such a logic as applied to linguistic utterances. In attempting to apply quantum logic to meaningful linguistic utterances, there is still a point that perhaps needs clarification. Specifically I refer to Grice's rules of cooperation in communication. We shall adopt Grice's rules as being both necessary and sufficient assumptions with which to properly interpret propositions. That is, instead of taking Grice's rules as being constraints on the speaker, we take them as being conditions of state in all meaningful utterances. Given Grice's rules, one can reasonably interpret any proposition in such a way as to identify the subspace of constraints and the relevant domain.

While this section will sometimes find the resolving of anomalies as produced by the application of classical logic to linguistic utterances a natural consequence of demonstrating the use of quantum logic, this is not the primary goal. We are well aware of the fact that many problems in the system will only be solved by further investigations, and indeed we will remain unaware of many of them until the bulk of classical logic laws have been reinvestigated without the use of distributive laws, distributive laws being the major modification which quantum logic proposes. We seek then, only to provide the reader with some relevant examples of the power of quantum logic in dealing with natural languages.

If indeed the logical substructure of natural languages is isomorphic with quantum logic, then we can reasonably expect to find macroscopic examples of microscopic quantum linguistic events which a classical logic will be unable to either predict or explain. In much the same way as macroscopic evidence for quantum processes can be found in physics by setting up the proper experimental conditions, we can also arrange linguistic circumstances in such a way as to observe evidence of quantum linguistic processes. Three classes of phenomena are suggested from considerations of the fundamental differences between quantum logic and classical logic. They are: first, that distributive laws are not generally valid; second, that properties are subject to a commutation relation; and three, that the ordering relation between properties need not be either linear or causal and must be defined in each specific case of the observation of linguistic data.

Distributive laws in an orthocomplemented modular lattice are satisfied only by simultaneously observable properties - that is, only if their individual domains are topologically equivalent.[28] The familiar distributive and commutative laws of adjunction and conjunction

P1 $\quad p \cup q = q \cup p$

P2 $\quad p \cap q = q \cap p$

P3 $\quad p \cap (q \cup r) = (p \cap q) \cup (p \cap r)$

P4 $\quad p \cup (q \cap r) = (p \cup q) \cap (p \cup r)$

are no longer to be taken as generally valid.therefore. Consider, for example, the permutation

1a) She became pregnant and married the child's father.

of the sentence

1b) She married the child's father and became pregnant.
The semantic interpretations of 1a and 1b are certainly not equivalent as the commutative law for conjunctions (P2) would have us believe. However, the fact is that the propositional variables as identified in the example are not commutative properties or simultaneously observable. They do not permute and thus P1 through P4 do not apply. Or consider the following argument which P2 and P3 would have us believe to be perfectly legitimate.

2) Mrs. Grundy passed away late yesterday afternoon and was either cremated or was buried at Forest Lawn. That is to say, either Mrs. Grundy passed away late yesterday afternoon and was cremated or Mrs. Grundy passed away late yesterday afternoon and was buried at Forest Lawn. Therefore Mrs. Grundy was cremated and passed away late yesterday afternoon or Mrs. Grundy was buried at Forest Lawn and passed away late yesterday afternoon. (Poor Mrs. Grundy!)

A classical logic evaluation of 1a and 1b, or 2 would force logical equivalence without taking into account the necessity for considering a commutation relation between the variables. Quantum logic not only demands the defining of such a relation, but predicts that the anomalies encountered will occur if the commutation relation is ignored.

The importance of the commutation relation, which results in a reevaluation of commutative and distributive laws, applies not only to the standard logical operators of conjunction, adjunction, orthocomplement, and inclusion, but to quantifiers and modal operators as well. This fact becomes quite apparent when the familiar universal and existential quantifiers are written as follows:

$$\forall x = x_1 \cap x_2 \cap x_3 \cap x_4 \cap \ldots\ldots\ldots\ldots x_n = \bigcap_x$$

and

$$\exists x = x_1 \cup x_2 \cup x_3 \cup x_4 \cup \ldots\ldots\ldots\ldots x_n = \bigcup_x$$

where n is the enumeration of the elements of an exhaustive denotation of the set of all x's. With these definitions it makes sense to speak of a distributive law involving quantifiers such as

$$(\forall x)(y \cup x) = (y \cup x_1) \cap (y \cup x_2) \cap (y \cup x_3) \cap \ldots\ldots (y \cup x_n)$$

We repeat, such laws are valid <u>only</u> where the variables permute.

One can reasonably expect a failure of classical logic whenever the variables in an argument do not permute. This situation will often occur where the causal ordering of events is quite obvious. However, variables may fail to permute in other ways such as by being complements and the only way to insure that variables are permutable is to define both the domain and the subspace of constraints. While on the subject of the commuting of variables, we would like to point out that the inclusive adjunction is a special case in which the atomic propositions happen to commute. We may, in fact, be well advised to take the exclusive adjunction as a warning that the atomic propositions are complementary - i.e. that exclusive adjunctions presuppose a complementarity relation between the adjuncts. Note that the definition of a variable plays an important role in determining how it may be used in logical arguments and specifically, what laws are valid in the argument.

A commutation relation defines a fundamental limit to observation. In the application of a quantum logic to linguistics such a limit will be in the nature of a limit on interpretation. In specifying a semantic space, one must recognize that the means for describing a semantic space must always be greatly exceeded by the ways in which semantic spaces can differ. We are, in fact, confronted with a fundamental ambiguity in natural language just as quantum logic would suggest. To put it simply, one can never communicate a semantic space with infinite precision. The pragmatic result is, of course, that meaningful linguistic utterances must always be given an <u>interpreted</u> meaning.

Such an interpretation can approximate the intended meaning only in as much as the ordering relation between quantum units of meaning is well-defined. The following examples make the importance of the ordering relation quite clear.

3a) I'll leave if and only if you have someone to take my place.
b) If I leave, you'll have someone to take my place, and if you have someone to take my place, I'll leave.
c) If you have someone to take my place, I'll leave, and if you don't have someone to take my place, I won't leave.
4a) My pulse goes above 100 if and only if I do heavy exercise.
b) If my pulse goes above 100, I do heavy exercise, and if I do heavy exercise, my pulse goes above 100.
c) If I do heavy exercise, my pulse goes above 100, and if I don't do heavy exercise, my pulse doesn't go above 100.
5a) Butter melts if and only if it is heated.
b) If butter melts, it is heated, and if butter is heated, it melts.
c) If butter is heated, it melts, and if butter is not heated, it doesn't melt.

These examples are all of the logical forms:

6a) A if and only if B.
b) If A then B, and if B, then A.
c) If B then A, and if not B, then not A.

Note that while examples of the form 6a) and 6c) are asymmetric, 6b) is symmetric. It presupposes a linear non-branching causal ordering relation which is not necessarily the case. The semantic interpretations of the propositions A and B serve to define the ordering relation and the form of statements such as 6b) is not sufficient in itself to provide such a definition. However, those of 6a) and 6c) do provide information about the ordering relation and the semantic interpretation of these is in fact more closely related.

Perhaps the most important result of quantum logic is that it predicts the difficulties encountered in what is known as either "fuzzy" or many-valued logic. McCawley (1975) has given many examples of the incoherent truth valuations which arise in attempting to assign multi-valued truth values to propositions. We can assume, on classical grounds, that no continuous multi-valued truth function can be erected within the propositional calculus in as much as it is a development from Boole's dual isomorphism. The nature of the algebra of all subsets of any complex assembly is obviously discontinous. Still, this does not tell us why we may not, for instance, state the truth value of "Superman is more heroic than Spider-man" in terms of some discrete truth function with 1 and 0 as upper and lower bounds respectively and increasing in increments of 1/n for n (finite) different possible values without contradicting the otherwise acceptable and desirable laws of logic. Quantum logic does give us the reason for this situation.

For a logic based on non-distributive lattices, the correspondence between propositions and truth values is not homomorphic except in the two-valued case. Thus the truth values of p and q do not necessarily determine the truth value of $p \supset q$. This fact alone is sufficient to account for the difficulties encountered in "fuzzy" logic. Based on this evidence we reject Lakoff's (1972) conclusion "that a multi-valued logic is essential for an adequate treatment of the semantics of a large amount of natural language vocabulary, particularly adjectives such as fat, obnoxious, and pleasant. Not only is it not essential but it is not even adequate! Such adjectives, which appear to be continuously variable in such a way as to force multi-valued logics, may in fact be analyzed in quite a different fashion.

Following the definition of a proposition set forth in the previous section, we claim that only the two-valued case has meaning. Given a domain a and a subspace of constraints A', the proposition A either is or is not contained in A' (remember that we are dealing with a quantum manifold). For example, "Jonathan Livingston Seagull is a happy bird" contains the adjective happy which, under Lakoff's analysis, would force the statement to have a degree of truth dependent upon the degree to which J.L. Seagull is a happy bird as divorced from being simply a bird. We claim instead that the subspace of constraints defines a single meaning for the adjective happy if the statement is in fact to be a proposition. Otherwise, the fundamental ambiguity of adjectives such as happy or jolly or obnoxious will be sufficient to insure that the statement is propositionally incomplete. Intuitively we know that while all words are ambiguous to some degree, under a given interpretation a statement is accepted as either true or false. But this points out another difficulty. Given two propositions which contain the same ambiguous adjectives, can a complex statement be made which contains both of them and which is subject to a coherent truth valuation. The answer is obvious: only if the domain and the subspace of constraints of the component propositions are mutually inclusive. If this is not the case, the result is precisely that encountered by McCawley (1975) wherein denotative examples of statements such as "All fat persons are jolly" contribute differently to the truth value of the universal statement. Of course, the sense in which one would say "Rockefeller is both fat and jolly" or "Ed Levi is both fat and jolly" need not be and probably would not be the same. When we demand that the eigenfunction of a statement such as "All fat persons are jolly" be single-valued, it is precisely for this reason. Remember that the universal quantifier $\forall$ may be defined as an exhaustive conjunction of the denotative examples. However, for the conjunction to be a proposition it must be propositionally complete - i.e. the domain and subspace of constraints must be the same for all the conjuncts. Similar arguments apply to the problem encountered with "Some fat persons are jolly."[29]

On the macroscopic level, a failure to include a complete semantic description of a semantic subspace, where the subspace is said to be propositionally incomplete if the eigenfunction is not single-valued, results in an ambiguity. This phenomenon is of a quantum nature and is perhaps the most common "anomaly." The phenomenon can be understood intuitively. Given a statement such as the famous Bach-Peters (1968) sentence (The pilot that shot at it hit the MIG that chased him.) which is ambiguous in the sense demonstrated by Kartunnen (1971), the most natural response would be to ask the speaker questions which would specify the pronoun referents. Quantum logic leads us to believethat the statement, as it stands, is not propositionally complete. It does not contain enough information for the eigenfunction to be single-valued and, in this case, the failure occurs in the subspace of constraints. Such ambiguities are common and in fact are the determining factor in triggering (intellectual?) conversation. Ambiguities need not be of such an overt linguistic nature, however, and can result from a lack of stress, pitch, expression, location, topic, membership, etc. Consider, for example the following.

7) Some men love all women.

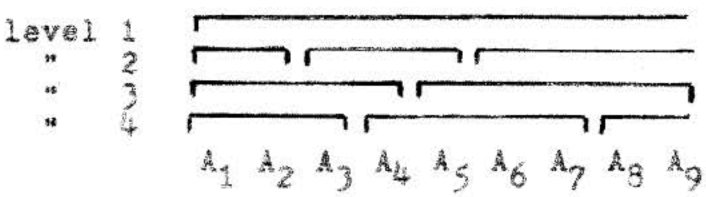
This is an example of phonemic ambiguity. The meaning can obviously be resolved through an application of stress; one case in which the final stress is placed on <u>all</u> and the other in which it is placed on <u>women</u>. Thus propositional completeness can be dependent upon stress.

We can not generally assume that propositional completeness is linear functional. For instance in the following examples the deletion of the n't of 8a and 8c produces a negation in one case (8b) and a completely different meaning in the second (8d).

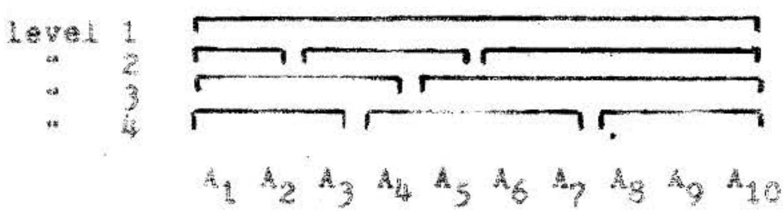
8a) John doesn't love his wife.
b) John does love his wife.
c) Some people aren't afraid of dying.
d) Some people are afraid of dying.

8c is an example of the previously mentioned superproposition. The semantic subspace of which it is composed is a set of mutually exclusive component propositions each of which becomes either propositionally complete or propositionally incomplete with the addition or deletion of some meaningful unit such as n't.

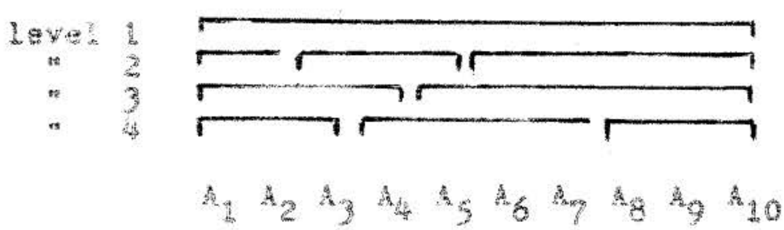
Quite often a linguistic utterance is composed of more than one syntagma and these may overlap each other as well as be linearly decomposable. With the addition of each new quantum unit of meaning (constraint), each of several relevant subspaces is "triggered" so to speak. A subspace is deleted when a digit is required which can not be part of the domain of the semantic space. A subspace is complete (closed) when no further addition may define it further or (in some cases) cause it to be deleted. Thus a string may have the following semantic structure:



Levels refer to different component propositions. The addition of A_{10} may change the structure as follows:



On the other hand, the following may occur:



- where two previously complete component propositions are now incomplete at two different levels. And, of course, any combination of loss of completeness or satisfying of completeness at different levels can occur.

Only complete propositions become relevant to the interpretation of the string and <u>each</u> additional semantic digit is capable of affecting this completeness at all semantic levels - and any position in the string may be the boundary of some semantic space.

A rather blatant example of a statement which is propositionally incomplete[30] was given by McCawley (1975):

9) The Battle of Hastings is Jewish.

A typical response to such a statement might well be "What?!" The statement is almost irrepairably incomplete and the extremes to which one would have to go in redefining the terms to make a meaningful utterance out of it would be unlikely in most natural circumstances.

Consider the information which can not be logically expressed in statements such as imperatives, declaratives, interrogatives, etc. These are statements which may qualify as propositions if the internal states of the speaker are included in the semantic content. For instance, "Open the door!" may be taken as a statement that there exists an internal desire for the door to be open, a desire for the addressee to perform the action indicated, and an implicit assumption that the desired action is possible. Interrogatives imply similar conditions except that there is a recognition of context, i.e. that the action may not be possible, and that there are social constraints on both parties. Declaratives may well be only an attempt to make internal constraints external and thus apply them to other individuals. Exclamatory statements relate many forms of internal emotional (and meaningful) subspace constraints and here stress, intonation, facial expression, etc. may provide the information to the addressee.

V. Conclusions

We are forced to conclude that the standard notion of truth valuations is primarily at fault when difficulties arise with applications of logic to linguistic data. The classical definition of a proposition allows truth valuations of objects which should not be considered propositions in the first place, and, secondly, does not explicitly show where the failure occurs. The approach presented here, that of using quantum logic, has produced a more restricted definition of a proposition which is, I believe, more intuitive as well. And, further, quantum logic has predicted that all attempts to find a coherent multi-valued truth valuation for a calculus relating to propositions of natural languages are doomed to failure, the difficulty arising from the fact that the truth values of p and q will not determine that of the proposition $p \supset q$. In short, the major statement which quantum logic makes about classical logic is that, where truth valuations are concerned, it falls flat on its face when applied to a corpus of linguistic data.

The second major modification which quantum logic proposes is with regard to the distributive and commutitive laws. The number of logical arguments which will have to be reevaluated by quantum logic represents a task of considerable size. And we did not attack the problem here, the purpose being rather to derive a method of doing so at a later time. It would not, however, be surprising if many of the problems encountered in classical logic proofs (such as the absurdity of being able to infer anything from a contradiction) were found to be examples of illegitimate arguments under the rules of inference allowed in quantum logic. What is generally important is that one can not simply plug in propositions in an argument. The validity of an argument in quantum logic is strictly dependent on the relative content of the propositions used in the argument, since the distributive or commutative laws can be used only where the propositions permute as defined earlier. Thus we are faced with a second major prediction of quantum logic: all attempts to provide a logical analysis of the semantic content of a linguistic utterance taken out of context are doomed to failure. In short, the linguists practice of considering utterances independent of context is faulty from start to finish. And the meaningful data of which that context is comprised is probably not even linguistic by standard definitions.

By no means are we saying that we have shown that the proper logical substructure of natural languages is the quantum logic presented here. However, the explanatory power of quantum logic does appear to be greater than that of classical logic. As such the subject certainly deserves more study. Several questions need to be explored. First, how does quantum logic change the standard proofs of completeness (axiomatic)? Second, how will arguments such as those presented by Gödel (undecidability) be changed? And finally, the mathematical arguments which were used in the development of the meta-language should be made explicit. For example, the assumption that meaning might be quantized was based on a combination of mathematical argument and empirical evidence with regard to the storage capacity of the human brain and these might be clarified in the future for those whose area of expertise is not mathematics.

We would like to note in conclusion that many of the concepts employed were derived with an eye to using them later in studying the "dynamics" of quantum linguistic logic - the generation and recognition of utterances. It is hoped that no unnecessary confusion resulted.

Footnotes

1. The second main conclusion of Gödels undecidability theorem states that there are an infinite number of unproveable but valid theorems in any axiomatic system sufficiently rich to contain arithmatic.

2. Consider counterfactuals. For examples see McCawley (1975)

3. We refer to any logic having as a basis the algebra of attributes proposed by Boole as a classical logic.

4. See the article by Ernst Nagel in <u>The World of Mathematics</u> vol.3.

5. To some extent this paper is concerned with precisely the question of linguitic determinism and relativity - therefore the Whorf Hypothesis. In as much as classical logic is based on the geometry of Euclid and quantum logic, more nearly on that of Einstein, Putnam has suggested that the logic is "warped". Hence the subtitle for which I am indebted to the humor of David McNeill, University of Chicago.

6. Note that Birkhoff hinted at this remarkable concept as early as 1940 in <u>Lattice Theory</u> in a section entitled "Critique of Boole's dual isomorphism" in which he notes that a "universe" is subject to change as a result of the state of one's knowledge.

7. The problem here is not with the definitions; that is, not with the geometry. Due to the concept of locality a line may both converge and be straight in the sense of Euclid. This is a change, not in our geometry, but in our way of thinking - i.e. in our logic.

8. I assume here that a single logical substructure exists. The sense in which I am using logic is to refer to any generative mechanism whereby explicit rules of inference may be specified and the

end-product is a set of statements subject to truth valuations. It is certainly counter-intuitive to assume that some such substructure does not exist and that utterances are "born full-fledged from the head of Zeus."

9. Boole's dual isomorphism (of which the propositional calculus is an example) leads to grave difficulties and Birkhoff sumarizes these in his critique.

10. see Finkelstein, "Space-Time Code" for an explicit account. Briefly, there appears to be evidence that the world is both discrete and continuous.

11. We are interested here in a model for "static" systems of linguistic logic. A relativistic model would inherently be concerned with the "dynamics" of such a system. The implications of relativistic quantum logic for the understanding of the learning, generation, and recognition of linguistic utterances will be elaborated in forthcoming papers.

12. ....and this under the assumption that the majority of linguistic utterances do admit to a causal logic.

13. see Finkelstein, STC II.

14. adapted from Finkelstein, STC II.

15. see Feynman, R.P. The Character of Physical Laws

16. "Words" seem to correspond to "strings" and the "STC" to the linguists concept of syntax.

17. Metric is used here in the sense of Fréchet.

18. see Birkhoff, G., Lattice Theory, especially with regard to von Neumann's continuous geometry.

20. One might call this a sort of "quantum tagmemics".

21. Note that the parentheticals of "p and (q and p)" may well serve to clarify the domains over which a given subspace of constraints operates. The explicit use of domains will in general force the above to be different from "p and q".

22. Quantifiers can sometimes serve to define the domain of a proposition. Notice, however, that the scope of quantifiers (in normal usuage) is context dependent. In particular, <u>most</u>, <u>many</u>, <u>few</u>, <u>several</u>, etc. show this dependency most readily. <u>Most</u> and <u>many</u> are sometimes equivalent quantifiers and sometimes not. Of particular interest are the cases in which the scope of <u>all</u>, <u>none</u>, <u>each</u>, and <u>any</u> are context sensitive. Quantifiers are not sufficient to define a domain.

23. The notion of non-degeneracy is simply explained in Born, <u>Atomic Physics</u>.

24. Logical completeness is a necessary but not sufficient condition for propositional completeness. As an example, consider the argument "If Germany had invaded England, it would have won the war. Therefore, if Germany hadn't invaded England, it wouldn't have won the war." Under most circumstances which would provide the context to produce propositional completeness, this would end by being false. Counterfactuals are often of this nature - only grammatically acceptable and logically complete (composed of wff's).

25. Note that context need not be of a purely linguistic nature to be meaningful.

26. i.e. is <u>degenerate</u>.

27. Proof: Consider the proposition $(R_n \subset P)_D$. Let there be n individuals with subspaces of constraint R_n defined on the domain D. By definition $R_1 = R_2 = R_3 = R_4 = \dots R_n = R$ and since the truth value

$/(R_n \subset P)_D/$ must be unique for all values of n and R_n=R, we have $/(R_n \subset P)_D/ = /(R \subset P)_D/$.

28. The topology of the semantic space is defined by the subspace of constraints as well as the "elements" of the domain itself.

29. Notice that universals of this sort may be treated also as being parametric component propositions in a superproposition. Then the problem is that universals are not necessarily linear truth functionals - i.e. the universal is a superproposition of component propositions with mutually exclusive domains and/or subspaces of constraint.

30. It should be noted that a so-called satisfiable proposition is, by its very nature, propositionally incomplete and as such is essentially a contradiction in terms. If it is neither "true" nor "false" in all states of affairs, a statement simply is not propositionally complete.

TABLE I

Simple Systems

For quantum systems, the algebra of a system S^A is irreducible being the algebra of all maps of an underlying innerproduct space $\underline{I}(S)$. In this part all concepts are relative to one implicit system S.

Class (of a system S): = projection (quantity equal to its * (adjoint) and square) in S^A; subspace P,Q,... of the underlying linear space of S^A.

$P \subset Q$, P is included in Q (of classes P, Q): = the basic eigenvalue equation $PQ=P$; the subspace inclusion $P \subset Q$.

$\underline{I}$ and $\emptyset$, universal and null class: = quantities 1 and 0; $\underline{I}(S)$ and the 0 vector, as subspaces of $\underline{I}(S)$.

$P \cup Q$, P or Q (adjunction): = $\sup(P,Q)$; span $P \cup Q$ (the set join of two subspaces never being required).

$P \cap Q$, P and Q (conjunction): = $\inf(P,Q)$; subspace meet $P \cap Q$.

Q is a complement of P: $=P \cup Q = \underline{I}, P \quad Q=\emptyset$; Q is a complementary subspace to P.

$-P$, the negation of P: $=1-P$; orthogonal complement of subspace P.

$P \perp Q$, P excludes Q: $=PQ=0$; P and Q are orthogonal subspaces.

P comQ, P is compatible or commutes with Q: a basis exists for $\underline{I}(S)$ adapted to both subspaces P and Q.

$f(S)$, a coordinate f of S: =map $f: S \rightarrow \underline{C}$; spectral family $dP_f(z)$ of subspaces, z a complex variable. Any coordinate f may be represented by a coordinate quantity $f=\int z \, dP_f(z)$, where the projection-valued measure $dP_f(z)$ is defined by the algebra map $f_A: \underline{C}^A \rightarrow S^A$.

$P \subset_1 Q$, P is <u>just included</u> in Q: $P \subset X \subset Q$ if and only if P=X or X=Q; Q=PU one additional 1-space.

$|P|$, the <u>measure</u> of P:= the length m of a chain $0 \subset_1 P_1 \subset_1 \ldots \subset_1 P_m = P$.

<u>o</u>, a <u>singlet</u>;= projection <u>o</u> with measure = 1; a ray or 1-space of <u>I</u>(S).

If G is any group of maps $g: S \to S$ and G_A is the group of induced algebra maps, we can then define as follows:

S/G, S over G: =the algebra $S^A \backslash G_A$, the collection of those quantities of S^A invariant under G_A; the algebra of operators on <u>I</u>(S) commuting with all members of the (unitary) group G. Even if S is a quantum system, S/G generally is not.

S G, S under G: = the algebra S^A/G_A resulting from S^A by identification with respect to G_A; the subspace of <u>I</u>(S)consisting of all fixed points under G.

Let P be a class of S:

$S \backslash P$. S <u>under</u> P, the restriction of S to P: = the algebra PS^AP taken with the +, X, * of S^A but with the new unit P; the subsystem defined by a subspace $P \subset$ <u>I</u>(S)

The system 1: = the system whose algebra is <u>C</u>; system with a one-dimensional Hilbert space. The system 1 is both a classical system (commutative) and a quantum system (irreducible).

In quantum logic the <u>distributive law</u> is weakened to the form

If $a \subset c$, then $a \cup (b \cap c) = (a \cup b) \cap c$. Note that it is self-dual: replacing $\subset, \cap, \cup$ by $\supset, \cup, \cap$ merely replaces a,b,c by c,b,a. It also follows that $(a \cup b) \cap c = (a \cup b) \cap (a \cup c)$.

For quantum assemblies, it is not generally true that $a \supset b = -a \cup b$.

Compound Systems

S+T, the sum of S and T: = the direct-product algebra $S^A T^A$, in which the two algebras S^A and T^A commute; the direct product Hilbert space $\underline{I}(S) X \underline{I}(T)$. Similarly for IIS_1. Associative and distributive laws hold.

S $\underline{R}$ T, a binary relation $\underline{R}$ between systems S,T: =a class of ST; subspace of the direct product $\underline{I}(S) X \underline{I}(T)$.

S-T, similar systems S,T: = two systems S,T provided with an equivalence map $e:S \rightarrow T$ (map with inverse); two Hilbert spaces with a unitary $e:\underline{I}(S) \rightarrow \underline{I}(T)$. We designate corresponding projections in S,T by P(S)-P(T). Replicas of a system S are similar systems obtained from S by attaching labels,e.g. S_1-S_2

S=T: = for similar systems S-T, the class $U_{\underline{a}}\underline{a}(S)\underline{a}(T)$, the union extending over all singlets $\underline{a}(S)$-$\underline{a}(T)$; symmetric subspace of the direct product.

Reflexive relation: =relation S $\underline{R}$ T with $(S=T) \subset (S\ \underline{R}\ T)$; subspace of $\underline{I}(S) X \underline{I}(T)$ including the symmetric subspace.

R^T, the transpose of R: $=eXe^{-1}(R)$ where $e:S \rightarrow T$ is the equivalence map of S-T and $\underline{R}$=S $\underline{R}$ T.

Symmetric relation: =relation $S=S^T$.

Transitive relation: =relation $\underline{T}$ with $S_1\ \underline{T}\ S_2, S_2\ \underline{T}\ S_3 \subset S_1\ \underline{T}\ S_3$

Functional relation: =relation S $\underline{F}$ T= $U_{\underline{a}}\underline{a}f_A(\underline{a})$,where $\underline{a}$ ranges over the singlets of S, and $f:S \rightarrow T$ is a map; the graph $U_{\underline{a}}(\underline{a} X f_A(\underline{a}))$ of an algebra map $f_A:T^A \rightarrow S^A$.

seq_2S, the 2-sequence of S's: =the product S_1S_2 of two replicas S_1-S_2 of S; the ordered pair of two S's.

dia_2S, the diagonal 2-sequence of S's: = $seq_2S \setminus (S_1=S_2)$, the restriction of S_1S_2 to the class $(S_1=S_2)$; the subspace of symmetric tensors in $\underline{I}(S_1) X \underline{I}(S_2)$

Let G be the symmetric group on two similar systems, S_1-S_2.

ser_2S, the <u>2-series of S</u>'s: $=seq_2S/G$ with G as above; the subalgebra of $S^A X S^A$ invariant under transposing; the direct sum of the subalgebras of the symmetric and antisymmetric subspace of $\underline{I}(S)X\underline{I}(S)$.

Complex Systems

seq_nS, the <u>n-sequence of S's</u>: $= \Pi_m S_m$ $(m=1,\ldots,n)$, where S_m-S are similar systems; the direct product of n replicas of $\underline{I}(S)$.

seqS, the <u>sequence of S's</u>:= ${}_n seq_n S$ (n=0,1,...);the Maxwell-Boltzman Fock space over $\underline{I}(S)$, with the number operator N as superselection rule.

dia_nS, the <u>diagonal n-sequence of S's</u>: $=seq_nS$ $(S_1=\ldots=S_n)$; the space of symmetric tensors of degree n over $\underline{I}(S)$.

diaS, the <u>diagonal sequence of S's</u>: = ${}_n dia_n S$ (n=0,1,....); the Bose-Einstein Fock space over $\underline{I}(S)$, with the number of systems N as superselection.

Let G be the symmetric group on the systems in a sequence seqS. Then we define as follows:

serS, the <u>series of S's</u>: =seqS/G; the subalbegra of $seqS^A$ invariant under G.

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